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An upwind difference scheme on a novel Shishkin-type mesh for a linear convection–diffusion problem☆

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Abstract

We consider an upwind finite difference scheme on a novel layer-adapted mesh (a modification of Shishkin's piecewise uniform mesh) for a model singularly perturbed convection–diffusion problem in two dimensions. We prove that the upwind scheme on the modified Shishkin mesh is first-order convergent in the discrete L^∞ norm, independently of the diffusion parameter ε, provided only that the perturbation parameter satisfies $\varepsilon \leqslant N^{-1}$, where $\mathcal{O}(N^2)$ mesh points are used. The new mesh yields more accurate results than simple upwinding on a standard Shishkin mesh, even though it requires essentially the same computational effort. Numerical experiments support these theoretical results. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper we consider the singularly perturbed boundary value problem

$$Lu := -\varepsilon \Delta u + b_1(x,y)u_x + b_2(x,y)u_y = f(x,y) \quad \text{on } \Omega = (0,1)^2, \tag{1a}$$

$$u = 0 \quad \text{on } \Gamma = \partial\Omega, \tag{1b}$$

where ε is a small positive parameter, $b_1(x,y) \geqslant \gamma\beta_1$ and $b_2(x,y) \geqslant \gamma\beta_2$ for all $(x,y) \in \bar{\Omega}$, where $\beta_1 > 0$, $\beta_2 > 0$ and $\gamma > 1$ are constants. We assume that b_1, b_2 and f are smooth. The solution u of (1) typically has exponential boundary layers at the sides $x = 1$ and $y = 1$ of Ω.

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For small values of ε, it is well known that standard numerical methods for (1) are unstable and fail to give accurate results. There is a vast literature dealing with numerical methods for convection–diffusion and associated problems; see [9] or [11] for a survey.

Here we shall analyse a simple upwind difference scheme on a modified Shishkin mesh: a Bakhvalov–Shishkin mesh. We shall show that it gives more accurate results than on standard Shishkin meshes. Bakhvalov–Shishkin meshes can be used whenever Shishkin meshes are applicable, but are easier to handle than pure Bakhvalov meshes — in particular when curved layers or interior layers are present. Madden and Stynes [7] demonstrate the use of Shishkin meshes in those cases.

Let a set of mesh nodes be given by $\Omega^N = \{(x_i, y_j)\colon i,j=0,\dots,N\}$ with $0=x_0 < x_1 < \cdots < x_N = 1$ and $0 = y_0 < y_1 < \cdots < y_N = 1$, where N, our discretisation parameter, is an even positive integer. We set $\Gamma^N = \Gamma \cap \Omega^N$. Given a function $\{v_{ij}\}$ defined on this mesh, we introduce the following difference operators:

$$D_x^- v_{ij} = \frac{v_{ij} - v_{i-1,j}}{h_{x;i}}, \quad D_y^- v_{ij} = \frac{v_{ij} - v_{i,j-1}}{h_{y;j}}, \quad D_x^+ v_{ij} = \frac{v_{i+1,j} - v_{ij}}{h_{x;i+1}},$$

$$D_y^- v_{ij} = \frac{v_{i,j+1} - v_{ij}}{h_{y;j+1}}, \quad D_x^0 v_{ij} = \frac{v_{i+1,j} - v_{i-1,j}}{h_{x;i+1} + h_{x;i}}, \quad D_y^0 v_{ij} = \frac{v_{i,j+1} - v_{i,j-1}}{h_{y;j+1} + h_{y;j}},$$

$$\delta_x^2 v_{ij} = \frac{2}{h_{x;i+1} + h_{x;i}}(D_x^+ v_{ij} - D_x^- v_{ij}) \quad \text{and} \quad \delta_y^2 v_{ij} = \frac{2}{h_{y;j+1} + h_{y;j}}(D_y^+ v_{ij} - D_y^- v_{ij}),$$

where $h_{x;i} = x_i - x_{i-1}$ and $h_{y;j} = y_j - y_{j-1}$ for $i,j = 1,\dots,N$. These operators are approximations of the first-order and second-order derivatives of v at (x_i, y_j); D^- is upwinded, while D^0 and δ^2 are central difference operators and D^+ is downwinded. To simplify the notation we set $g_{ij} = g(x_i, y_j)$ for any function g, while g_{ij}^N denotes an approximation of g at the point (x_i, y_j).

For small ε the use of D^0 on uniform meshes in approximating the first-order derivatives leads to nonphysical oscillations in the computed solution. This is due to a loss in stability — unless the mesh diameter is extremely small, which is computationally expensive. Upwind schemes do not have this disadvantageous property. For example, the simple upwind scheme

$$L_u^N u_{ij}^N := (-\varepsilon(\delta_x^2 + \delta_y^2) + b_{1;ij} D_x^- + b_{2;ij} D_y^-) u_{ij}^N = f_{ij} \quad \text{on } \Omega^N \setminus \Gamma^N, \tag{2a}$$

$$u_{ij}^N = 0 \quad \text{on } \Gamma^N \tag{2b}$$

is stable and oscillation free, for its associated matrix is an M-matrix. On a standard Shishkin mesh (see Section 3) this method is almost first-order convergent, uniformly with respect to the perturbation parameter ε; more precisely, the error in the computed solution satisfies [5, Remark 3.3]

$$|u_{ij} - u_{ij}^N| \leqslant \begin{cases} CN^{-1} & \text{for } 0 \leqslant i,j \leqslant N/2, \\ CN^{-1} \ln N & \text{otherwise} \end{cases} \tag{3}$$

with a constant C that is independent of ε and N.

We shall see that the same difference scheme on a modified Shishkin mesh that incorporates an idea by Bakhvalov [2] is first-order convergent, uniformly in the perturbation parameter, and therefore yields more accurate results than on a standard Shishkin mesh.

An outline of the paper is as follows: in Section 2 we state some properties of the exact solution. Based on these results we introduce a modified Shishkin mesh in Section 3. In Section 4 we analyse the convergence properties of the scheme. Finally, numerical results are presented in Section 5. There we also compare the performance of the classical central difference scheme on Shishkin's original mesh and on the Bakhvalov–Shishkin mesh.

Notation. Throughout the paper, C will denote a generic positive constant (possibly subscripted) that is independent of ε and of the mesh. Note that C is not necessarily the same at each occurrence.

2. Properties of the exact solution

To construct layer-adapted meshes correctly, it is crucial to have a precise knowledge of the asymptotic behaviour of the exact solution. The decomposition of the solution of (1) in Lemma 1 provides that information. It is also the key to proving the main results in Section 4.

Let

$$\mathscr{L}_i v := \frac{\partial v}{\partial y}\frac{\partial^i}{\partial x^i}\left(\frac{b_2}{b_1}\right) \quad \text{for } i = 0, 1.$$

Lemma 1 (Shishkin-type decomposition). *Let the functions b_1 and b_2 be smooth. Let $f \in C^{4,\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1]$, where $C^{4,\alpha}(\bar{\Omega})$ is the classical Hölder space of functions whose derivatives up to the fourth order are Hölder continuous with exponent α. Suppose that f satisfies the compatibility conditions*

$$f(0,0) = f(0,1) = f(1,1) = f(1,0) = 0,$$

that

$$\left(\frac{f}{b_1}\right)_y (0,0) = \left(\frac{f}{b_2}\right)_x (0,0),$$

$$\left(\left(\frac{f}{b_1}\right)_x - \mathscr{L}_0\left(\frac{f}{b_1}\right)\right)_y (0,0) = \left(\frac{f}{b_2}\right)_{xx} (0,0),$$

$$\left(\left(\frac{f}{b_1}\right)_{xx} - \mathscr{L}_0\left(\left(\frac{f}{b_1}\right)_x - \mathscr{L}_0\left(\frac{f}{b_1}\right)\right) - 2\mathscr{L}_1\left(\frac{f}{b_1}\right)\right)_y (0,0) = \left(\frac{f}{b_2}\right)_{xxx} (0,0)$$

and

$$\left(b_2\left(\frac{f}{b_2}\right)_{xx}\right)(0,0) = \left(b_1\left(\frac{f}{b_1}\right)_{yy}\right)(0,0).$$

Let $\|\cdot\|_k$ denote the standard supremum norm in $C^k(\Omega)$. Then the boundary value problem (1) *has a classical solution $u \in C^{3,1}(\bar{\Omega})$, and this solution can be decomposed as*

$$u = S + E_1 + E_2 + E_{12},$$

where for all $(x, y) \in \bar{\Omega}$ *we have*

$$\|S\|_2 \leqslant C,$$
$$\left|\frac{\partial^{i+j} E_1}{\partial x^i \partial y^i}(x, y)\right| \leqslant C\varepsilon^{-i} \mathrm{e}^{-\beta_1(1-x)/\varepsilon},$$
$$\left|\frac{\partial^{i+j} E_1}{\partial x^i \partial y^i}(x, y)\right| \leqslant C\varepsilon^{-j} \mathrm{e}^{-\beta_2(1-y)/\varepsilon}$$

and

$$\left|\frac{\partial^{i+j} E_{12}}{\partial x^i \partial y^i}(x, y)\right| \leqslant C\varepsilon^{-(i+j)} \mathrm{e}^{-(\beta_1(1-x)+\beta_2(1-y))/\varepsilon}$$

for $0 \leqslant i + j \leqslant 2$. *Furthermore*,

$$|LE_1(x, y)| \leqslant C\mathrm{e}^{-\beta_1(1-x)/\varepsilon},$$
$$|LE_2(x, y)| \leqslant C\mathrm{e}^{-\beta_2(1-y)/\varepsilon}$$

and

$$|LE_{12}(x, y)| \leqslant C\mathrm{e}^{-\beta_1(1-x)/\varepsilon}\mathrm{e}^{-\beta_2(1-y)/\varepsilon}.$$

Proof. See [6]. □

Remark 2. While the conditions of Lemma 1 look restrictive, they are used in the present paper only to obtain the conclusions of the lemma. In our later error analysis we require only the bounds on the derivatives of S and the functions E.

3. Layer-adapted meshes

The construction of our layer-adapted meshes is based on the decomposition of the previous section.

3.1. Shishkin mesh

Standard Shishkin meshes [8,12] are piecewise uniform meshes, constructed a priori, that are refined in the layers. We briefly describe their construction.

Let $\lambda_{S,x}$ and $\lambda_{S,y}$ denote two mesh transition parameters defined by

$$\lambda_{S,x} = \min\left(\frac{1}{2}, \frac{\varepsilon}{\beta_1} \ln N\right) \quad \text{and} \quad \lambda_{S,y} = \min\left(\frac{1}{2}, \frac{\varepsilon}{\beta_2} \ln N\right). \tag{5}$$

The mesh transition parameters have been chosen so that the boundary layer terms in the asymptotic expansion of u (the terms E_1, E_2 and E_{12} above) are of order N^{-1} on $\Omega_{S,s} = (0, 1-\lambda_{S,x}] \times (0, 1-\lambda_{S,y}]$. Now $[0, 1-\lambda_{S,x}]$ and $[1-\lambda_{S,x}]$ are uniformly divided into $N/2$ subintervals.

We refer the reader to [8,10] for a detailed discussion of their properties and applications.

3.2. Bakhvalov–Shishkin mesh

Our new mesh is a modification of the Shishkin mesh discribed above that incorporates an idea by Bakhvalov [2], who proposed a mesh condensing in the boundary layers by effectively inverting the boundary layer terms. But the original Bakhvalov mesh requires the solution of a nonlinear equation to determine the transition point where the mesh switches from coarse to fine. Instead we fix the transition points as in the Shishkin mesh. We introduce mesh parameters $\lambda_{B,x}$ and $\lambda_{B,y}$ as follows:

$$\lambda_{B,x} = \min\left(\frac{1}{2}, \frac{2\varepsilon}{\beta_1}\ln N\right) \quad \text{and} \quad \lambda_{B,y} = \min\left(\frac{1}{2}, \frac{2\varepsilon}{\beta_2}\ln N\right). \tag{6}$$

Assumption 3. We now make the very mild assumption that $\lambda_x = (2\varepsilon/\beta_1)\ln N$ and $\lambda_y = (2\varepsilon/\beta_2)\ln N$, as otherwise N^{-1} is exponentially small compared with ε. We shall also assume throughout the paper that $\varepsilon \leqslant N^{-1}$ as is generally the case in practice.

Now the interval $[0, 1-\lambda_{B,x}]$ is uniformly dissected into $N/2$ subintervals, while $[1-\lambda_{B,x}, 1]$ is partitioned into the same number of mesh intervals by inverting the function $\exp(-\beta_1(1-x)/(2\varepsilon))$. We specify the x_i, $i = N/2, \ldots, N$, so that $\{e^{-\beta_1(1-x_i)/(2\varepsilon)}\}_i$ is a linear function in i, i.e., we set

$$e^{-\beta_1(1-x_i)/(2\varepsilon)} = Ai + B$$

and determine the unknowns A and B such that $x_{N/2} = 1-\lambda_{B,x}$ and $x_N = 1$. This gives

$$x_i = \begin{cases} \left(1 - \dfrac{2\varepsilon}{\beta_1}\ln N\right)\dfrac{2i}{N} & \text{for } i = 0, \ldots, N/2, \\[2ex] 1 + \dfrac{2\varepsilon}{\beta_1}\ln\left(\dfrac{N^2 - 2(N-i)(N-1)}{N^2}\right) & \text{for } i = N/2+1, \ldots, N \end{cases}$$

with an analogous formula for the mesh points y_j.

Remark 4. In contrast to Bakhvalov-type meshes, the underlying mesh generating function of this new mesh is not $C^1[0,1]$, but only $C^0[0,1]$ and it is dependent on the number of mesh points used.

The following lemma gives some estimates of the mesh sizes that will be used later.

Lemma 5. *The step sizes of the mesh Ω^N satisfy*

$$h_{x;i} \leqslant 2N^{-1} \quad \text{and} \quad h_{x;N/2+i} \leqslant \frac{4\varepsilon}{\beta_1 i} \leqslant CN^{-1}$$

for $i = 1, \ldots, N/2$. Analogous estimates hold true for the $h_{y;j}$.

Proof. The first estimate is trivial. Recalling the definition of the mesh we obtain for the step size

$$h_{x;N/2+i} = \frac{2\varepsilon}{\beta_1}\ln\left(\frac{(2i+1)N - 2i}{(2i-1)N - 2(i-1)}\right) \leqslant \frac{2\varepsilon}{\beta_1}\ln\frac{2i+1}{2i-1} \quad \text{for } i = 1, 2 \ldots . N/2. \tag{7}$$

For $i \geqslant 1$ we have $(2i-1)e^{2/i} \geqslant (2i-1)(1+2/i) = 2i+3-2/i \geqslant 2i+1$. Thus

$$\ln \frac{2i+1}{2i-1} \leqslant \frac{2}{i}.$$

We apply this estimate to (7) and recall $\varepsilon \leqslant N^{-1}$ to complete the proof. □

4. Analysis of the scheme on a Bakhvalov–Shiskin mesh

In this section we give an analysis of the error of the simple upwind scheme (2) on the Bakhvalov–Shishkin mesh defined in Section 3. The analysis is based on the discrete comparison principle and barrier function technique introduced by Kellogg and Tsan [4], which was recently applied to a hybrid difference scheme on a Shishkin mesh in two dimensions in [5].

For our analysis we shall assume that the conclusions of Lemma 1 hold true.

We start by stating two lemmas that can be verified by easy computations. The matrix associated with L_u^N is an M-matrix. Therefore the following lemma holds true.

Lemma 6 (Discrete comparison principle). *The operator L_u^N satisfies a discrete comparison principle*, i.e., *if $\{v_{ij}\}$ and $\{w_{ij}\}$ are two mesh functions satisfying $|v_{ij}| \leqslant w_{ij}$ on Γ^N and $|L_u^N v_{ij}| \leqslant L_u^N w_{ij}$ on $\Omega^N \backslash \Gamma^N$, then $|v_{ij}| \leqslant w_{ij}$ on Ω^N.*

Remark 7. In Lemma 6, we say that w_{ij} is a *barrier function* for v_{ij}.

This comparison principle plays a fundamental role in our analysis. We shall compute a certain bound for the truncation error $\{L_u^N(u-u^N)_{ij}\}$, construct a suitable barrier function $\{w_{ij}\}$ and apply Lemma 3 to bound the error $\{(u-u^N)_{ij}\}$.

We shall use the next lemma to bound the truncation error. It can be proved by using Taylor's formula with the integral form of the remainder. For convenience we set $L_{u;x}^N = -\varepsilon\delta_x^2 + b_{1;ij}D_x^-$ and $L_{u;y}^N = -\varepsilon\delta_y^2 + b_{2;ij}D_y^-$. The continuous operator can be split in an analogous fashion: $L = L_x + L_y$.

Lemma 8 (Truncation error). *Let $g(x,y)$ be a smooth function defined on Ω. Then the following estimates for the truncation error hold true*:

$$|L_{u;x}^N g_{ij} - (L_x g)_{ij}| \leqslant C(\varepsilon + h_{x;i} + h_{x;i+1}) \max_{\xi \in [x_{i-1}, x_{i+1}]} \left| \frac{\partial^2 g}{\partial x^2}(\xi, y_j) \right|,$$

and

$$|L_{u;x}^N g_{ij} - (L_x g)_{ij}| \leqslant C \left\{ \varepsilon \int_{x_{i-1}}^{x_{i+1}} \left| \frac{\partial^3 g}{\partial x^3}(\xi, y_j) \right| \mathrm{d}\xi + \int_{x_i-1}^{x_i} \left| \frac{\partial^2 g}{\partial x^2}(\xi, y_j) \right| \mathrm{d}\xi \right\}$$

for $0 < i,\ j < N$, with analogous estimates for $|L_y^N g_{ij} - (L_y g)_{ij}|$.

Recalling the decomposition of Lemma 1, we split the solution of our discrete problem in a similar manner. We define S^N, E_1^N, E_2^N and E_{12}^N by

$$L_u^N S_{ij}^N = (LS)_{ij}, \qquad L_u^N E_{1;ij}^N = (LE_1)_{ij},$$
$$L_u^N E_{2;ij}^N = (LE_2)_{ij}, \qquad L_u^N E_{12;ij}^N = (LE_{12})_{ij} \quad \text{on } \Omega^N \setminus \Gamma^N, \tag{8a}$$

and

$$S_{ij}^N = S_{ij}, \qquad E_{1;ij}^N = E_{1;ij}, \qquad E_{2;ij}^N = E_{2;ij}, \qquad E_{12;ij}^N = E_{12;ij} \quad \text{on } \Gamma^N. \tag{8b}$$

Then the error can be split:

$$|u_{ij} - u_{ij}^N| \leqslant |S_{ij} - S_{ij}^N| + |E_{1;ij} - E_{1;ij}^N| + |E_{2;ij} - E_{2;ij}^N| + |E_{12;ij} - E_{12;ij}^N|.$$

The various terms on the right-hand side are dealt with separately.

Regular part of the solution: Lemmas 1, 5 and 8 yield

$$|L_u^N(S_{ij} - S_{ij}^N)| = |L_u^N S_{ij} - (LS)_{ij}| \leqslant CN^{-1} \quad \text{on } \Omega^N \setminus \Gamma^N.$$

Choosing the barrier function $w_{ij} = C_0 N^{-1} x_i$ (with the constant C_0 sufficiently large), we see from Lemma 6 that

$$|S_{ij} - S_{ij}^N| \leqslant w_{ij} \leqslant CN^{-1} \quad \text{on } \Omega^N. \tag{9}$$

Boundary layer parts: Imitating [4], we set

$$G_{ij} = \prod_{k=i+1}^{N} \left(1 + \frac{\beta_1 h_{x;k}}{2\varepsilon}\right)^{-1} \quad \text{on } \Omega^N,$$

with the convention that $G_{Nj} = 1$ for $j = 0, \dots, N$. This mesh function is a discrete equivalent of the function $\exp(-\beta_1(1-x)/(2\varepsilon))$.

Lemma 9. *The mesh function G_{ij} satisfies*

$$L_u^N G_{ij} \geqslant C_1 G_{ij} / \max\{h_{x;i}, \varepsilon\} \tag{10}$$

on $\Omega^N \setminus \Gamma^N$, for some positive constant C_1, and

$$\mathrm{e}^{-\beta_1(1-x_i)/(2\varepsilon)} \leqslant G_{ij}. \tag{11}$$

Proof. This is an easy calculation. See also [4]. □

Set $w_{ij} = C_2 G_{ij}$; we shall show that C_2 can be chosen so that w_{ij} is a barrier function for $E_{1;ij}^N$. First, from Lemma 1 and (8a) we see that

$$|L_u^N E_{1;ij}^N| = |LE_1(x_i, y)| \leqslant C\mathrm{e}^{-\beta_1(1-x_i)/\varepsilon} \leqslant C\mathrm{e}^{-\beta_1(1-x_i)/(2\varepsilon)} \quad \text{on } \Omega^N \setminus \Gamma^N. \tag{12}$$

It follows from (10), (11) and (12) that (when C_2 is sufficiently large), for $0 < i,\ j < N$ we have

$$L_u^N w_{ij} \geqslant C_1 C_2 G_{ij} / \max\{h_{x;i}, \varepsilon\} \geqslant |L_u^N E_{1;ij}^N|. \tag{13}$$

On Γ^N we have

$$|E_{1;ij}^N| = |E_{1;ij}| \leqslant C\mathrm{e}^{-\beta_1(1-x_i)/\varepsilon} \leqslant CG_{ij}.$$

Thus we can choose C_2 so that w_{ij} is a barrier function for $E^N_{1;ij}$. Hence, using Lemma 1, we obtain the estimate

$$|E^N_{1;ij} - E_{1;ij}| \leqslant |E^N_{1;ij}| + |E_{1;ij}| \leqslant w_{ij} + C\mathrm{e}^{-\beta_1(1-x_i)/\varepsilon}.$$

From (11) we conclude that

$$|E^N_{1;ij} - E_{1;ij}| \leqslant CG_{ij} \quad \text{on } \Omega^N. \tag{14}$$

Next, we show $G_{ij} \leqslant CN^{-1}$ for $i = 0,\dots,N/2$. It is sufficient to show that $G_{N/2,j} \leqslant CN^{-1}$ since $G_{ij} \leqslant G_{i+1,j}$ for $0 \leqslant i < N$ and $0 \leqslant j \leqslant N$. We have

$$\ln\left(\prod_{k=N/2+i}^{N}\left(1+\frac{\beta_1 h_{x;k}}{2\varepsilon}\right)\right) \geqslant \sum_{k=N/2+1}^{N}\left(\frac{\beta_1 h_{x;k}}{2\varepsilon} - \frac{1}{2}\left(\frac{\beta_1 h_{x;k}}{2\varepsilon}\right)^2\right)$$

$$\geqslant \frac{\beta_1 - x_{N/2}}{2\varepsilon} - 2\sum_{k=1}^{N/2} k^{-2} \geqslant \frac{\beta_1 - x_{N/2}}{2\varepsilon} - \frac{\pi^2}{3},$$

where we have used Lemma 5. Thus,

$$G_{N/2,j} = \prod_{k=N/2+i}^{N}\left(1+\frac{\beta_1 h_{x;k}}{2\varepsilon}\right)^{-1} \leqslant C\mathrm{e}^{-\beta_1(1-x_{N/2})/(2\varepsilon)}.$$

Recalling that $x_{N/2} = 1 - 2\varepsilon/\beta_1 \ln N$, we obtain from (14)

$$|E_{1,ij} - E^N_{1,ij}| \leqslant C\prod_{k=N/2+1}^{N}\left(1+\frac{\beta_1 h_{x;k}}{2\varepsilon}\right)^{-1} \leqslant CN^{-1} \quad \text{for } 0 \leqslant i \leqslant N/2 \quad \text{and} \quad 0 \leqslant j \leqslant N. \tag{15}$$

We use a consistency and barrier function argument to estimate $|E^N_{1;ij} - E_{1;ij}|$ on $\Omega^N_b = \Omega^N \cap ([1-\lambda_{B,x}] \times [0,1])$. We have, by (15),

$$|E^N_{1;ij} - E_{1;ij}| \leqslant CN^{-1} \quad \text{on } \Gamma^N_b = \Omega^N_b \cap \partial([1-\lambda_{B,x}] \times [0,1]). \tag{16}$$

For $N/2 < i < N$ and $0 < j < N$, Lemmas 1, 5 and 8 yield

$$|L^N_u(E^N_{1;ij} - E_{1;ij})| \leqslant C\left\{N^{-1} + \int_{x_{i-1}}^{x_{i+1}} \varepsilon^{-2}\mathrm{e}^{-\beta_1(1-x)/\varepsilon}\,\mathrm{d}x\right\}.$$

Let $x = \varphi(\xi) = 1 + (2\varepsilon/\beta_1)\ln[1 - 2(1-\xi)(1-N^{-1})]$ and $\xi_i = \varphi^{-1}(x_i)$. Then

$$|L^N_u(E^N_{1;ij} - E_{1;ij})| \leqslant C\{N^{-1} + \varepsilon^{-1}\int_{\xi_{i-1}}^{\xi_{i+1}} \mathrm{e}^{-\beta_1(1-\varphi(\xi))/(2\varepsilon)}(1-N^{-1})\,\mathrm{d}\xi\}$$

$$\leqslant C\{N^{-1} + \varepsilon^{-1}N^{-1}\mathrm{e}^{-\beta_1(1-x_{i+1})/(2\varepsilon)}\}, \quad \text{since } \xi_{i+1} - \xi_{i-1} = 2N^{-1}$$

$$= C\{N^{-1} + \varepsilon^{-1}N^{-1}\mathrm{e}^{-\beta_1(1-x_i)/(2\varepsilon)}\mathrm{e}^{\beta_1 h_{x;i+1}/(2\varepsilon)}\}$$

$$\leqslant C\{N^{-1} + \varepsilon^{-1}N^{-1}\mathrm{e}^{-\beta_1(1-x_i)/(2\varepsilon)}\},$$

since $h_{x;i+1} \leqslant 4\varepsilon/\beta_1$ for $N/2 \leqslant i < N$, see Lemma 5. Now an application of Lemma 9 yields

$$|L^N_u(E^N_{1;ij} - E_{1;ij})| \leqslant CN^{-1}(1 + \varepsilon^{-1}G_{ij}) \leqslant CN^{-1}L^N_u(x_i + G_{ij}), \tag{17}$$

using $h_{x;i} \leqslant 4\varepsilon/\beta_1$ for $N/2 < i \leqslant N$ again.

The discrete comparision principle of Lemma 6 holds true also for L_u^N on the smaller mesh domain Ω_b^N. This principle is applied using (16) and (17) and the barrier function

$$w_{ij} = C_3 N^{-1}(1 + x_i + G_{ij}),$$

where C_3 is chosen sufficiently large. Then Lemma 6 gives

$$|E_{1;ij}^N - E_{1;ij}| \leqslant CN^{-1} \text{ on } \Omega_b^N.$$

Recalling estimate (15) we have

$$|E_{1;ij}^N - E_{1;ij}| \leqslant CN^{-1} \quad \text{for } 0 \leqslant i,\ j \leqslant N. \tag{18}$$

We clearly have an analogous result for the other boundary layer:

$$|E_{2;ij}^N - E_{2;ij}| \leqslant CN^{-1} \quad \text{for } 0 \leqslant i,\ j \leqslant N. \tag{19}$$

Corner layer part: The procedure for estimating $|E_{12}^N - E_{12}|$ is similar to that for the boundary layer parts, but the mesh function

$$\bar{G}_{ij} = \prod_{k=i+1}^{N} \left(1 + \frac{\beta_1 h_{x;k}}{2\varepsilon}\right)^{-1} \prod_{l=j+1}^{N} \left(1 + \frac{\beta_2 h_{y;l}}{2\varepsilon}\right)^{-1} \quad \text{on } \Omega^N$$

is used to bound the error.

First we get, similarly to (15),

$$|E_{12;ij}^N - E_{12;ij}| \leqslant CN^{-1} \quad \text{on } \Omega^N \setminus \Omega_c^N, \tag{20}$$

where we have set $\Omega_c = (1 - \lambda_{B,x}) \times (1 - \lambda_{B,y})$ and $\Omega_c^N = \Omega^N \cap \Omega_c$.

Finally $|E_{12;ij}^N - E_{12;ij}|$ has to be estimated on Ω_c^N. Inequalities (20) and (8b) imply that

$$|E_{12;ij}^N - E_{12;ij}| \leqslant CN^{-1} \quad \text{on } \Gamma_c^N = \Gamma^N \cap \bar{\Omega}_c.$$

Estimates of the truncation error are provided by Lemmas 1 and 8. Hence,

$$\begin{aligned} |L_u^N(E_{12;ij}^N - E_{12;ij})| &\leqslant C\varepsilon^{-2}\left\{\int_{x_{i-1}}^{x_{i+1}} \mathrm{e}^{-(\beta_1(1-x)+\beta_2(1-y))/\varepsilon}\,\mathrm{d}x + \int_{y_{j-1}}^{y_{j+1}} \mathrm{e}^{-(\beta_1(1-x)+\beta_2(1-y))/\varepsilon}\,\mathrm{d}y\right\} \\ &\leqslant CN^{-1}\varepsilon^{-1}\bar{G}_{ij}, \end{aligned}$$

similarly to (17). Now the barrier function $w_{ij} = C_4 N^{-1}\{1 + \bar{G}_{ij}\}$ is used to bound the error on Ω_c^N. Thus,

$$|E_{12;ij}^N - E_{12;ij}| \leqslant CN^{-1} \quad \text{for } 0 \leqslant i,\ j \leqslant N. \tag{21}$$

Collecting estimates (9), (18), (19) and (21), we can state the main result of this paper.

Theorem 10. *Assume that the conclusions of Lemma* 1 *hold true. Then the error of the simple upwind scheme on the Bakhvalov–Shishkin mesh satisfies*

$$|u_{ij}^N - u_{ij}| \leqslant CN^{-1} \quad \textit{for } 0 \leqslant i,\ j \leqslant N.$$

5. Numerical results

In this Section we verify experimentally the theoretical results of Section 4. We compare the results obtained by the simple upwind scheme on standard Shishkin meshes and Bakhvalov–Shishkin meshes. As well as the simple upwind scheme, we also test the performance of the central difference scheme:

$$(-\varepsilon(\delta_x^2+\delta_y^2)+b_{1;ij}D_x^0+b_{2;ij}D_y^0)u_{ij}^N = f_{ij} \quad \text{on } \Omega^N\backslash\Gamma^N, u_{ij}^N = 0 \quad \text{on } \Gamma^N.$$

Example 11. We test the performance of the two schemes when applied to the boundary value problem

$$-\varepsilon\Delta u + 2u_x + 3u_y = f \quad \text{on } \Omega, u = 0 \quad \text{on } \Gamma,$$

where the right-hand side f is chosen so that

$$u = 2\sin x(1-\mathrm{e}^{-2(1-x)/\varepsilon})y^2(1-\mathrm{e}^{-3(1-y)/\varepsilon})$$

is the exact solution. It exhibits typical boundary layer behaviour — see Section 2.

The error of the schemes is measured in the discrete maximum norm. It depends on the perturbation parameter ε and the discritisation parameter N:

$$\mathrm{e}^{\varepsilon,N} = \max_{ij} |u_{ij}^N - u_{ij}|.$$

In our tests the ε-uniform error e^N and the corresponding convergence rates are estimated by

$$e^N = \max_{r=2,\dots,8} e^{10^{-r},N} \quad \text{and} \quad p^N = \frac{\ln e^N - \ln e^{2N}}{\ln 2}.$$

Table 1 displays the error and the corresponding convergence rates of the simple upwind scheme on the two meshes. It illustrates the difference between the error estimates of Theorem 1 and of (3). The use of Bakhvalov–Shishkin meshes produces more accurate results. In contrast to the original Shishkin mesh the convergence is not spoiled by a logarithmic factor.

Table 2 displays our experimental results for the central difference scheme. It too demonstrates the superiority of the new mesh: for $N=128$ the approximation is about four times better than that obtained on a standard Shishkin mesh. We observe second order convergence, while the convergence on the standard Shishkin mesh is slower — presumably only $\mathcal{O}N^{-2}\ln^2 N$; cf. [1] where central differencing on a standard Shishkin mesh for a two-point boundary value problem is studied.

Table 1
Simple upwinding for Example 11

Standard Shishkin mesh			Bakhvalov–Shishkin mesh		
N	e^N	p^N	N	e^N	p^N
32	1.205e-1	0.755	32	1.253e-1	0.952
64	7.143e-2	0.822	64	6.474e-2	1.004
128	4.039e-2	0.850	128	3.228e-2	1.015
256	2.241e-2	0.864	256	1.597e-2	1.014
512	1.232e-2	—	512	7.911e-3	—

Table 2
Central differencing for Example 11

Standard Shishkin mesh			Bakhvalov–Shishkin mesh		
N	e^N	p^N	N	e^N	p^N
8	1.030e-1	1.317	8	8.976e-2	1.835
16	4.135e-2	1.472	16	2.516e-2	1.957
32	1.491e-2	1.512	32	6.479e-3	1.957
64	5.227e-3	1.579	64	1.669e-3	1.990
128	1.749e-3	—	128	4.201e-4	—

Table 3
Simple upwinding for Example 12

Standard Shishkin mesh			Bakhvalov–Shishkin mesh		
N	e^N	p^N	N	e^N	p^N
32	3.656e-2	0.628	32	3.486e-2	0.787
64	2.365e-2	0.782	64	2.020e-2	0.914
128	1.376e-2	0.854	128	1.072e-2	0.966
256	7.610e-3	0.897	256	5.491e-3	0.981
512	4.088e-3	—	512	2.782e-3	—

Example 12. We now consider a problem whose solution is less smooth:

$$-\varepsilon\Delta u + (3-x)u_x + (2-y)u_y = 4y\sin(x)\cos(3y) \quad \text{on } \Omega,$$
$$u = 0 \quad \text{on } \Gamma.$$

Theorem 3.2 of [3] shows that the solution u lies only in $C^{1,1}(\bar{\Omega})$. Therefore, the existence of the Shishkin-type decomposition of Lemma 1 is not guaranteed for this test problem. The exact solution of this problem is not available. We therefore estimate the errors in our computed solutions by means of a higher-order method on the same mesh. We have used the streamline diffusion FEM with bilinear elements. Table 3 shows that the simple upwind scheme on the Bakhvalov–Shishkin mesh gives more accurate results than the same scheme on a standard Shishkin mesh.

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